Stable bundles on 3-fold hypersurfaces

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Abstract

Using monads, we construct a large class of stable bundles of rank 2 and 3 on 3-fold hypersurfaces, and study the set of all possible Chern classes of stable vector bundles.

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1 Introduction

Perhaps the most popular method of constructing rank 2 bundles over a 3-dimensional projective variety X is the so-called Serre construction. Given a local complete intersection, Cohen-Macaulay curve $C \subset X$, let $\mathcal{I}_C$ denote its ideal sheaf. Then consider the extension

$$0 \to \mathcal{O}_X \to E \to \mathcal{I}_C(k) \to 0 \quad .$$

Under some conditions on C, the rank 2 sheaf E is locally-free; moreover, C is the zero-scheme of a section in $H^0(E)$; see [5] for a detailed description. In some sense, every rank 2 bundle can be obtained in this way.

In this letter, we explore a different technique: monads. We were motivated by a preliminary version of a paper by Douglas, Reinbacher and Yau, who proposed, based on physical grounds, the following stronger version of the Bogomolov inequality [4, Conjecture 2.1]:

Conjecture. *Let X be a non-singular, simply-connected, compact Kähler manifold of dimension n, with Kähler class H. Assume that X has trivial or ample canonical bundle. If E is a H-stable holomorphic vector bundle over X of rank $r \geq 2$, then its Chern classes $c_1(E)$ and $c_2(E)$ satisfy the following inequality:*

$$\Delta(E) = \frac{1}{r^2}\left(2rc_2(E) - (r-1)c_1(E)^2\right) \cdot H^{n-2} \geq \frac{1}{12}c_2(TX) \cdot H^{n-2} \qquad (1)$$

We show that this conjecture cannot be true by providing examples of stable bundles of rank 2 and 3 that do not satisfy (1) on hypersurfaces of degree 4, 5 and 6 within $\mathbb{P}^4$. These counter-examples are obtained as special cases of a more general construction of stable rank 2 and 3 bundles over 3-fold hypersurfaces, see Theorem 2 below.

This conjecture was withdrawn in a revised version of the preprint, and the counter-examples here presented do not bear directly on the truth or falsity of the other conjectures in the revised version of [4]. These interesting conjectures, which provide sufficient conditions for the existence of stable bundles with given Chern classes, still stand.

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2 Hypersurfaces and monads on hypersurfaces

Let us begin by recalling some standard facts about hypersurfaces within complex projective spaces.

A hypersurface $X_{(d,n)} \hookrightarrow \mathbb{P}^n$ ($n \geq 4$) of degree $d \geq 1$ is the zero locus of a section $\sigma \in H^0(\mathcal{O}_{\mathbb{P}^n}(d))$; for generic σ, its zero locus is non-singular. It follows from the Lefschetz hyperplane theorem that every hypersurface is simply-connected and has cyclic Picard group [2]. It is also easy to see that hypersurfaces are arithmetically Cohen-Macaulay, that is $H^p(\mathcal{O}_{X_{(d,n)}}(k)) = 0$ for $1 \leq p \leq n-1$ and all $k \in \mathbb{Z}$. Finally, the restriction of the Kähler $\tilde{H}$ class of $\mathbb{P}^n$ induces a Kähler class H on $X_{(d,n)}$, which is the ample generator of $\mathrm{Pic}(X_{(d,n)})$. One can show that:

$$c_1(TX_{(d,n)}) = (n+1-d) \cdot H \quad \text{and}$$

$$c_2(TX_{(d,n)}) = \left(d^2 - (n+1)d + \frac{1}{2}n(n+1)\right) \cdot H^2 \quad . \tag{2}$$

In summary, hypersurfaces within $\mathbb{P}^n$ with $n \geq 4$ (and in fact any complete intersection variety of dimension at least 3) do satisfy all the conditions in the Conjecture.

Fixed an ample invertible sheaf $\mathcal{L}$ with $c_1(\mathcal{L}) = H$ on a projective variety V of dimension n, recall that the slope $\mu(E)$ with respect to $\mathcal{L}$ of a torsion-free sheaf E on $X_{(d,n)}$ is defined as follows:

$$\mu(E) := \frac{c_1(E) \cdot H^{n-1}}{rk(E)} \quad .$$

We say that E is stable with respect to $\mathcal{L}$ if for every coherent subsheaf $0 \neq F \hookrightarrow E$ with $0 < rk(F) < rk(E)$ we have $\mu(F) < \mu(E)$. In the case at hand, stability will always be measured in relation to the line bundle $\mathcal{O}_X(1)$ on the hypersurface $X_{(d,n)}$, whose first Chern class, denoted by H, is the ample generator of $\mathrm{Pic}(X_{(d,n)})$.

A *linear monad* on $X_{(d,n)}$ is a complex of holomorphic bundles of the form:

$$0 \to \mathcal{O}_X(-1)^{\oplus a} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus b} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad , \tag{3}$$

which is exact on the first and last terms. In other words, α is injective and β is surjective as bundle maps, and $\beta\alpha = 0$. The holomorphic bundle $E = \ker\beta/\mathrm{Im}\alpha$ is called the cohomology of the monad. Note that

$$\mathrm{ch}(E) = b - a \cdot \mathrm{ch}(\mathcal{O}_X(-1)) - c \cdot \mathrm{ch}(\mathcal{O}_X(1)) \quad .$$

In particular,

$$\mathrm{rk}(E) = b-a-c \ , \ \ c_1(E) = (a-c)\cdot H \ \text{ and } \ c_2(E) = \frac{1}{2}(a^2-2ac+c^2+a+c)\cdot H^2 \ , \tag{4}$$

where in this case $H = c_1(\mathcal{O}_X(1))$. The left hand side of (1) is given by:

$$\Delta(E) = \frac{1}{r^2}\left(2rc_2(E) - (r-1)c_1(E)^2\right) \cdot H^{n-2} = \frac{b(a+c) - 4ac}{(b-a-c)^2} \quad . \tag{5}$$

We will also be interested in the kernel bundle $K = \ker\beta$; it has the following topological invariants:

$$\mathrm{rk}(K) = b - c \ \ , \ \ c_1(K) = -c \cdot H \ \text{ and } \ c_2(K) = \frac{1}{2}(c^2 + c) \cdot H^2 \quad . \tag{6}$$

The left hand side of (1) is given by:

$$\Delta(K) = \frac{1}{r^2}\left(2rc_2(K) - (r-1)c_1(K)^2\right) \cdot H^{n-2} = \frac{bc}{(b-c)^2} \quad . \tag{7}$$

More on linear monads and their cohomology bundles can be found at [1, 7, 8, 9] and the references therein. Let us just mention a very useful existence theorem due to Fløystad in the case of projective spaces, but easily generalizable to hypersurfaces. Below, Fløystad's original result [3, Main Theorem] is adapted to fit our needs; the proof will not be given here, since we explicitly establish the existence of the linear monads used in this letter.

Theorem 1. *Let $X_{(d,n)}$ be a non-singular hypersurface of degree d within $\mathbb{P}^n$, $n \geq 4$. There exists a linear monad on X as in (3) if and only if*

- $b \geq a + c + n - 2$, *if n is odd;*
- $b \geq a + c + n - 1$, *if n is even*

Our counter-examples to Conjecture 1 will be constructed as kernel and cohomologies of linear monads over hypersurfaces. In order to establish their stability, we will need the following result:

Theorem 2. *Let V be a 3-dimensional non-singular projective variety with $\mathrm{Pic}(V) = \mathbb{Z}$, and consider the following linear monad:*

$$0 \to \mathcal{O}_V(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_V^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_V(1)^{\oplus c} \to 0 \qquad (c \geq 1) \tag{8}$$

1. *the kernel* $K = \ker\beta$ *is a stable rank* $2+c$ *bundle with* $c_1(K) = -c$ *and* $c_2(K) = \frac{1}{2}(c^2+c)$*;*

2. *the cohomology* $E = \ker\beta/\mathrm{Im}\alpha$ *is a stable rank* 2 *bundle with* $c_1(E) = 0$ *and* $c_2(E) = c$*.*

In Section 3 below we present our counter-examples, which arise as are special cases of Theorem 2. The existence of monads of the form (8) above for V being a 3-fold hypersurface is explicitly established in Section 4. The proof of Theorem 2 is left to Section 5.

3 Counter-examples

Following the notation in the previous section, set $X = X_{(d,4)}$ and let $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a basis of $H^0(\mathcal{O}_X(1))$. Consider the following linear monad on :

$$0 \to \mathcal{O}_X(-1) \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 4} \xrightarrow{\beta} \mathcal{O}_X(1) \to 0 \tag{9}$$

with maps given by:

$$\alpha = \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{pmatrix} \text{ and } \beta = \begin{pmatrix} -\sigma_2 & \sigma_1 & -\sigma_4 & \sigma_3 \end{pmatrix} .$$

It is easy to see that (9) is indeed a linear monad. We will show that for $d = 4, 5, 6$, either its kernel bundle or its cohomology bundle will provide counter-examples to Conjecture 1.

3.1 Sextic within $\mathbb{P}^4$

Let $X = X_{(6,4)}$ be a degree 6 hypersurface within $\mathbb{P}^4$; notice that $\omega_X = \mathcal{O}_X(1)$, so that X has ample canonical bundle. One easily computes that $c_2(X) = 16 \cdot H^2$.

By Theorem 2, the cohomology of the monad (9) is a stable rank 2 bundle with $c_1 = 0$ and $c_2 = 1$. One has that

$$\Delta(E) = \frac{1}{r^2}\left(2rc_2(E) - (r-1)c_1(E)^2\right) \cdot H = H^3$$

while

$$\frac{1}{12}c_2(TX) \cdot H = \frac{4}{3} \cdot H^3 \quad .$$

Therefore, the strong Bogomolov inequality (1) is not satisfied.

3.2 Quartic within $\mathbb{P}^4$

Let $X = X_{(4,4)}$ be a degree 4 hypersurface within $\mathbb{P}^4$; notice that $\omega_X = \mathcal{O}_X(-1)$, so that X has ample anti-canonical bundle. One easily computes that $c_2(X) = 6 \cdot H^2$.

By Theorem 2, the kernel bundle of the monad (9) is a stable rank 3 bundle with $c_1 = -1$ and $c_2 = 1$. One has that

$$\Delta(K) = \frac{1}{r^2}\left(2rc_2(K) - (r-1)c_1(K)^2\right) \cdot H = \frac{4}{9} \cdot H^3$$

while

$$\frac{r^2}{12} c_2(TX) \cdot H = \frac{1}{2} \cdot H^3 \quad .$$

Therefore, the strong Bogomolov inequality (1) is not satisfied.

3.3 Quintic within $\mathbb{P}^4$

Let $X = X_{(5,4)}$ be a degree 5 hypersurface within $\mathbb{P}^4$; notice that $\omega_X = \mathcal{O}_X$, so that X has trivial canonical bundle. One easily computes that $c_2(X) = 10 \cdot H^2$.

By Theorem 2, the kernel bundle of the monad (9) is a stable rank 3 bundle with $c_1 = -1$ and $c_2 = 1$. One has that

$$\Delta(K) = \frac{1}{r^2}\left(2rc_2(K) - (r-1)c_1(K)^2\right) \cdot H = \frac{4}{9} \cdot H^3$$

while

$$\frac{r^2}{12} c_2(TX) \cdot H = \frac{5}{6} \cdot H^3 \quad .$$

Therefore, the strong Bogomolov inequality (1) is not satisfied.

3.4 Is it possible to strengthen the Bogomolov inequality?

It is actually impossible to have an inequality of the form

$$\Delta(E) = \frac{1}{r^2}\left(2rc_2(E) - (r-1)c_1(E)^2\right) \cdot H^{n-2} \geq \kappa c_2(TX) \cdot H^{n-2} \qquad (10)$$

where E is a stable bundle and κ some constant, if the underlying variety is allowed to be too general.

Indeed, as it follows from Theorem 2 and the construction of Section 4, given a 3-fold hypersurface $X = X_{(d,4)}$, one can always find, for each $c \geq 1$, a stable rank $2+c$ bundle $K \to X$ with $c_1(K) = -c$ and $c_2(K) = (c^2+c)/2$, so that:

$$\Delta(K) = \frac{(2+2c)c}{(2+c)^2} \quad .$$

Notice that the minimum value for $\Delta(K)$ is $4/9$, which occurs exactly for $c = 1$. On the other hand, by formula (2), the right hand side of (10) grows quadratically with the degree d.

Therefore in order for an inequality of the form (10) to hold one must somehow restrict the type of varieties allowed, e.g. one could take only Fano and/or Calabi-Yau varieties.

3.5 Chern classes of stable rank 2 bundles on 3-fold hypersurfaces

The characterization of all possible cohomology classes that arise as Chern classes of stable bundles on a given Kähler manifold is not only of mathematical interest, but it is also relevant from the point of view of physics: it amounts to describing the set of all possible charges of BPS particles in type IIa superstring theory.

The integral cohomology ring of a 3-fold hypersurface $X = X_{(d,4)}$ is simple to describe:

$$H^*(X,\mathbb{Z}) = \mathbb{Z}[H,L,T]/(L^2 = T^2 = 0, H^2 = dL, HL = T) \quad .$$

Notice that $H^3 = dT$ and $H^4 = 0$. Clearly, H is the generator of $H^2(X,\mathbb{Z})$, L is the generator of $H^4(X,\mathbb{Z})$ and T is the generator of $H^6(X,\mathbb{Z})$.

Now let E be a rank r bundle on X. Recall that for any rank r bundle E on a variety X with cyclic Picard group, there is a uniquely determined integer k_E such that $-r+1 \leq c_1(E(k_E)) \leq 0$; the twisted bundle $E_{\mathrm{norm}} = E(k_E)$ is called the *normalization* of E. Therefore it is enough to consider the case when $c_1(E) = k \cdot H$ for $-r+1 \leq k \leq 0$, and study the sets $S_{(r,k)}(X)$ consisting of all integers $\gamma \in \mathbb{Z}$ for with there exists a stable rank r bundle E with $c_1(E) = k \cdot H$ and $c_2(E) = \gamma \cdot L$.

In the simplest possible case, provided by $d = 1$ (so that $X = \mathbb{P}^3$) and $r = 2$, this problem was completely solved by Hartshorne in [5]. He proved that $S_{(2,0)}(\mathbb{P}^3)$ consists of all positive integers, while $S_{(2,-1)}(\mathbb{P}^3)$ consists of all positive even integers. As far as it is known to the author, Hartshorne's result has not been generalized for other 3-folds.

As a consequence of Theorem 2, we have:

Lemma 3. *For every positive integer $c \geq 1$, $cd \in S_{(2,0)}(X_{(d,4)})$.*

Based on Hartshorne's result mentioned above, it seems reasonable to conjecture that $S_{(2,0)}(X_{(d,4)})$ consists exactly of all positive multiples of d.

The monad construction does not yield stable rank 2 bundles with odd first Chern class; to construct those, one needs a variation of the Serre construction, which provides a 1-1 correspondence between rank 2 bundles and codimension 2 subvarieties on $\mathbb{P}^3$; see Hartshorne's paper [5].

4 Existence of linear monads on 3-fold hypersurfaces

Let $X = X_{(d,4)}$ be a hypersurface of degree d within $\mathbb{P}^4$; as above let $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\}$ be a basis of $H^0(\mathcal{O}_X(1))$. We will now explicitly establish the existence of linear monads of the form

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_V^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad (c \geq 1) \quad .$$

Consider the $c \times (c+1)$ matrices:

$$B_1 = \begin{pmatrix} \sigma_1 & \sigma_2 & & & \\ & \sigma_1 & \sigma_2 & & \\ & & \ddots & \ddots & \\ & & & \sigma_1 & \sigma_2 \end{pmatrix} \quad B_2 = \begin{pmatrix} \sigma_3 & \sigma_4 & & & \\ & \sigma_3 & \sigma_4 & & \\ & & \ddots & \ddots & \\ & & & \sigma_3 & \sigma_4 \end{pmatrix} ,$$

and the $(c+1) \times c$ matrices:

$$A_1 = \begin{pmatrix} \sigma_2 & & & \\ \sigma_1 & \sigma_2 & & \\ & \ddots & \ddots & \\ & & \sigma_1 & \sigma_2 \\ & & & \sigma_1 \end{pmatrix} \quad A_2 = \begin{pmatrix} \sigma_4 & & & \\ \sigma_3 & \sigma_4 & & \\ & \ddots & \ddots & \\ & & \sigma_3 & \sigma_4 \\ & & & \sigma_3 \end{pmatrix} ,$$

Notice that all four matrices have maximal rank c. It easy to check that:

$$B_1 A_2 = B_2 A_1 = \begin{pmatrix} \phi_1 & \phi_2 & & & & \\ & \phi_0 & \phi_1 & \phi_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & & \phi_0 & \phi_1 \end{pmatrix} ,$$

where $\phi_0 = \sigma_1\sigma_3$, $\phi_1 = \sigma_1\sigma_4 + \sigma_2\sigma_3$ and $\phi_2 = \sigma_2\sigma_4$.

Now form the linear monad:

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0$$

where the maps α and β are given by:

$$\beta = \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \quad \text{and} \quad \alpha = \begin{pmatrix} A_2 \\ -A_1 \end{pmatrix}$$

Clearly, both maps are of maximal rank c for every point in X, and $\beta\alpha = B_1A_2 - B_2A_1 = 0$.

5 Proof of Theorem 2

The proof is based on a very useful criterion (due to Hoppe) to decide whether a bundle on a variety with cyclic Picard group is stable. We set $E_{norm} := E(k_E)$ and we call E normalized if $E = E_{\mathrm{norm}}$. We then have the following criterion.

Proposition 4. *([6, Lemma 2.6]) Let E be a rank r holomorphic vector bundle on a variety X with Pic$(X) = \mathbb{Z}$. If $H^0((\wedge^q E)_{\mathrm{norm}}) = 0$ for $1 \leq q \leq r-1$, then E is stable.*

Our argument follows [1, Theorem 2.8]. Consider the linear monad

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad ;$$

setting $K = \ker \beta$; one has the sequences:

$$0 \to K \to \mathcal{O}_X^{\oplus 2+2c} \xrightarrow{\beta} \mathcal{O}_X(1)^{\oplus c} \to 0 \quad \text{and} \tag{11}$$

$$0 \to \mathcal{O}_X(-1)^{\oplus c} \xrightarrow{\alpha} K \to E \to 0 \quad . \tag{12}$$

First, we will show that the kernel bundle K is stable. That implies that K is simple, which in turn implies that cohomology bundle E is simple Since any simple rank 2 bundle is stable, we conclude that E is also stable.

Recall that one can associate to the short exact sequence of locally-free sheaves $0 \to A \to B \to C \to 0$ two long exact sequences of symmetric and exterior powers:

$$0 \to \wedge^q A \to \wedge^q B \to \wedge^{q-1} A \otimes C \to \cdots \to B \otimes S^{q-1} C \to S^q C \to 0 \tag{13}$$

$$0 \to S^q A \to S^{q-1} A \otimes B \to \cdots \to A \otimes \wedge^{q-1} B \to \wedge^q B \to \wedge^q C \to 0 \tag{14}$$

In what follows, $\mu(F) = c_1(F)/\mathrm{rk}(F)$ is the slope of the sheaf F, as usual.

Finally, notice that $H^p(\mathcal{O}_X(k)) = 0$ for $p \geq 2$ and $k \geq -1$, by the Kodaira vanishing theorem.

Claim. *K is stable.*

From the sequence dual to sequence (11), we get that:

$$\mu(K^*) = \frac{c}{c+2} \quad \Longrightarrow \quad \mu(\wedge^q K^*) = \frac{qc}{c+2}$$

so that $(\wedge^q K^*)_{\mathrm{norm}} = \wedge^q K^*(k)$ for some $k \leq -1$, and if $H^0(\wedge^q K^*(-1)) = 0$, then $H^0((\wedge^q K^*)_{\mathrm{norm}}) = 0$.

The vanishing of $h^0(K^*(-1))$ (i.e $q = 1$) is obvious from the dual to sequence (11). For the case $q = 2$, start from the dual to (11) and consider the associated sequence

$$0 \to S^2(\mathcal{O}_X(-1)^{\oplus c}) \to \mathcal{O}_X(-1)^{\oplus c} \otimes \mathcal{O}_X^{\oplus 2c+2} \to \wedge^2(\mathcal{O}_X^{\oplus 2c+2}) \to \wedge^2 K^* \to 0 \quad .$$

Twist it by $\mathcal{O}_X(-1)$ and break it into two short exact sequences:

$$0 \to \mathcal{O}_X(-3)^{\oplus \binom{c+1}{2}} \to \mathcal{O}_X(-2)^{\oplus 2c^2+2c} \to Q \to 0$$

$$0 \to Q \to \mathcal{O}_X(-1)^{\oplus \binom{2c+2}{2}} \to \wedge^2 K^*(-1) \to 0$$

Passing to cohomology, we get $H^0(\wedge^2 K^*(-1)) = H^1(Q) = 0$.

Now set $q = 3 + t$ for $t = 0, 1, \cdots, c - 2$ and note that

$$\mu(\wedge^{3+t} K^*(-t-1)) = \frac{(3+t)c}{c+2} - t - 1 = 2\frac{c-t-1}{c+2} > 0 \quad .$$

Thus $(\wedge^{3+t} K^*)_{\mathrm{norm}} = \wedge^{3+t} K^*(k)$ for some $k \leq -t-2$, and if $H^0(\wedge^{3+t} K^*(-t-2)) = 0$, then $H^0((\wedge^{3+t} K^*)_{\mathrm{norm}}) = 0$.

We show that $H^0(\wedge^{3+t} K^*(-t-2)) = 0$ by induction on t. From the dual to sequence (12) we get, after twisting by $\mathcal{O}_X(-2)$:

$$0 \to \wedge^3 K^*(-2) \to \wedge^2 K^*(-1)^{\oplus c} \to \cdots$$

since $\wedge^3 E^* = 0$ because E has rank 2. Passing to cohomology, we get that $H^0(\wedge^3 K^*(-2)) = 0$, since, as we have seen above, $H^0(\wedge^2 K^*(-1)) = 0$. This proves the statement for $t = 0$.

By the same token, we get from the dual to sequence (12) after twisting by $\mathcal{O}_X(-2-t)$:

$$0 \to \wedge^{3+t} K^*(-2-t) \to \wedge^{2+t} K^*(-t-1)^{\oplus c} \to \cdots \quad .$$

Passing to cohomology, we get

$$H^0(\wedge^{2+t} K^*(-t-1)) = 0 \quad \Rightarrow \quad H^0(\wedge^{3+t} K^*(-t-2)) = 0$$

which is the induction step we needed.

In summary, we have shown that $H^0((\wedge^q K^*)_{\mathrm{norm}}) = 0$ for $1 \leq q \leq c+1$, thus by (4) we complete the proof of the claim.

Claim. *E is simple, hence stable.*

Tensoring by E the sequence dual to (12) we get

$$0 \to H^0(E^* \otimes E) \to H^0(K^* \otimes E) \to \cdots \quad . \tag{15}$$

Now tensoring (12) by K^* we get:

$$H^0(K^*(-1))^{\oplus c} \to H^0(K^* \otimes K) \to H^0(K^* \otimes E) \to H^1(K^*(-1))^{\oplus c} \quad .$$

But it follows from the dual of sequence (11) twisted by $\mathcal{O}_X(-1)$ that $h^0(K^*(-1)) = h^1(K^*(-1)) = 0$; thus $h^0(E^* \otimes E) = 1$ because K is simple. But E has rank 2, thus E is stable, as desired.

This completes the proof of the Theorem 2. □

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